# Learning to Play Two-Player Perfect-Information Games without Knowledge

Quentin Cohen-Solal<sup>a</sup>

<sup>a</sup>CRIL, Univ. Artois and CNRS, F-62300 Lens, France LAMSADE, Université Paris-Dauphine, PSL, CNRS, France

#### Abstract

In this paper, several techniques for learning game state evaluation functions by reinforcement are proposed. The first is a generalization of tree bootstrapping (tree learning): it is adapted to the context of reinforcement learning without knowledge based on non-linear functions. With this technique, no information is lost during the reinforcement learning process. The second is a modification of minimax with unbounded depth extending the best sequences of actions to the terminal states. This modified search is intended to be used during the learning process. The third is to replace the classic gain of a game (+1/-1) with a *reinforcement heuristic*. We study particular reinforcement heuristics such as: quick wins and slow defeats; scoring; mobility or presence. The fourth is another variant of unbounded minimax, which plays the safest action instead of playing the best action. This modified search is intended to be used after the learning process. The fifth is a new action selection distribution. The conducted experiments suggest that these techniques improve the level of play. We apply these different techniques to design program-players to the game of Hex (size 11 and 13) surpassing the level of Mohex 3HNN with reinforcement learning from self-play without knowledge. Keywords: Machine Learning, Tree Search, Games, Game Tree Learning, Minimax, Heuristic, Action Distribution, Unbound Minimax, Hex.

URL: cohen-solal@cril.fr (Quentin Cohen-Solal)

# 1. Introduction

One of the most difficult tasks in artificial intelligence is the *sequential decision making problem* [1], whose applications include robotics and games. As for games, the successes are numerous. Machine surpasses man for several games, such as backgammon, checkers, chess, and go [2]. A major class of games is the set of two-player games in which players play in turn, without any chance or hidden information. This class is sometimes called two-player *perfect information*<sup>1</sup> games Mycielski [3] or also two-player combinatorial games. There are still many challenges for these games. For example, for the *game of Hex*, computers have only been able to beat strong humans since 2020 [4]. For *general game playing* [5] (even restricted to games with perfect information): man is always superior to machine on an unknown complex game (when man and machine have a relatively short learning time to master the rules of the game). In this article, we focus on two-player zero-sum games with perfect information, although most of the contributions in this article should be applicable or easily adaptable to a more general framework<sup>2</sup>.

The first approaches used to design a game-playing program are based on a game tree search algorithm, such as *minimax*, combined with a handcrafted game state evaluation function based on expert knowledge. A notable use of this technique is the Deep Blue chess program [8]. However, the success of Deep Blue is largely due to the raw power of the computer, which could analyze two hundred million game states per

<sup>&</sup>lt;sup>1</sup>With some definitions, perfect information games include games with chance. It depends on whether one includes information about the future in perfect knowledge.

<sup>&</sup>lt;sup>2</sup>All the proposed techniques are directly applicable to the framework of perfect information single-agent sequential decision-making problems and in particular to solitaire games with perfect information. Tree learning, ordinal distribution, and reinforcement heuristics are applicable in the generalized cases (in case of m agents, reinforcement heuristics can be valued in real m-space). Minimax variants are applicable in multi-player perfect-information games as variants of the paranoid algorithm [6]. Descent can be applied to learn value functions for stochastic perfect-information games by applying it to a large set of determinist games corresponding to the stochastic game where the chance was fixed and known before the start of the game. To use the corresponding learned heuristic, either a minimax at depth 1 or an algorithm such as expectiminimax [7] must be used. However, for stochastic games and multiplayer games generalizations of the minimax variants should give better results in general.

second. In addition, this approach is limited by having to design an evaluation function manually (at least partially). This design is a very complex task, which must, in addition, be carried out for each different game. Several works have thus focused on the automatic learning of evaluation functions [9]. One of the first successes of learning evaluation functions is on the Backgammon game [10]. However, for many games, such as Hex or Go, minimax-based approaches, with or without machine learning, have failed to overcome human. Two causes have been identified [11]. Firstly, the very large number of possible actions at each game state prevents an exhaustive search at a significant depth (the game can only be anticipated a few turns in advance). Secondly, for these games, no sufficiently powerful evaluation function could be identified. An alternative approach to solve these two problems has been proposed, giving notably good results to Hex and Go, called Monte Carlo Tree Search and denoted MCTS [12, 13]. This algorithm explores the game tree non-uniformly, which is a solution to the problem of the very large number of actions. In addition, it evaluates the game states from victory statistics of a large number of random end-game simulations. It does not need an evaluation function. This was not enough, however, to go beyond the level of human players. Several variants of Monte Carlo tree search were then proposed, using in particular knowledge to guide the exploration of the game tree and/or random end-game simulations [13]. Recent improvements in Monte Carlo tree research have focused on the automatic learning of MCTS knowledge and their uses. This knowledge was first generated by supervised learning [14, 15, 16, 17, 18] then by supervised learning followed by *reinforcement learning* [19], and finally by only reinforcement learning [2, 20, 21]. This allowed programs to reach and surpass the level of world champion at the game of Go with the latest versions of the program AlphaGo [19, 2]. In particular, AlphaGo zero [2], which only uses reinforcement learning, did not need any knowledge to reach its level of play. This last success, however, required 29 million games of self-play (with 1,600 state evaluations per move). This approach has also been applied to chess [22]. The resulting program broke the best chess program (which is based on minimax). AlphaZero was subsequently reimplemented in Polygames [4], an opensource and multi-game program, which was notably applied to Hex and Havannah. In particular, at the 2020 Computer Olympiad, the gold medals for these games were won by Polygames.

It is therefore questionable whether minimax is totally out of date or whether the spectacular successes of recent programs are more based on reinforcement learning than Monte Carlo tree search. In particular, it is interesting to ask whether reinforcement learning would enhance minimax enough to make it competitive with Monte Carlo tree search on games where it dominates minimax so far, such as Go or Hex.

In this article<sup>3</sup>, we therefore focus on reinforcement learning within the minimax framework. We propose and asses new techniques for reinforcement learning of evaluation functions. Then, we apply them to design new program-players to the game of Hex (without using other knowledge than the rules of the game). We compare this program-player to Mohex 3HNN [16], the best Hex program, champion at Hex (size 11 and 13) of the 2018 Computer Olympiad [24].

In the next section, we briefly present the game algorithms and in particular minimax with unbounded depth on which we base several of our experiments. We also present reinforcement learning in games, the game of Hex, the state of the art of game programs on this game, as well as other games on which experiments are performed. In the following sections, we propose different techniques aimed at improving learning performances and we expose the experiments carried out using these techniques. In particular, in Section 3, we extends the tree bootstrapping (tree learning) technique to the context of reinforcement learning without knowledge based on non-linear functions. In Section 4, we present a new search algorithm, a variant of unbounded minimax called *descent*, intended to be used during the learning process. In Section 5, we introduce reinforcement heuristics. Their usage is a simple way to use general or dedicated knowledge in reinforcement learning processes. We study several reinforcement heuristics in the context of different games. In Section 6, we propose another variant of unbounded minimax, which plays the safest action instead of playing the best action. This modified search is intended to be used after the learning process. In Section 7, we introduce a new action selection distribution and we apply it with all the previous techniques to design program-players to the game of Hex (size 11 and 13) and com-

<sup>&</sup>lt;sup>3</sup>Note that this paper is an extended, improved, and english version of [23].

pare them to Mohex 3HNN. Finally, in the last section, we conclude and expose the different research perspectives.

#### 2. Background and Related Work

In this section, we briefly present game tree search algorithms, reinforcement learning in the context of games and their applications to Hex and Chess (for more details about game algorithms, see [25]).

Games can be represented by their *game tree* (a node corresponds to a game state and the children of a node are the states that can be reached by an action). From this representation, we can determine the action to play using a game tree search algorithm. In order to win, each player tries to maximize his score (i.e. the value of the game state for this player at the end of the game). As we place ourselves in the context of two-player zero-sum games, to maximize the score of a player is to minimize the score of his opponent (the score of a player is the negation of the score of his opponent).

#### 2.1. Game Tree Search Algorithms

The central algorithm is *minimax* which recursively determines the value of a node from the value of its children and the functions min and max, up to a limit recursion depth. With this algorithm, the game tree is uniformly explored. A better implementation of minimax uses *alpha-beta pruning* [26, 25] which makes it possible not to explore the sections of the game tree which are less interesting given the values of the nodes already met and the properties of min and max. Many variants and improvements of minimax have been proposed [27]. For instance, *iterative deepening* [28, 29] allows one to use minimax with a time limit. It sequentially performs increasing depth alpha-beta searches as long as there is time. It is generally combined with the *move ordering* technique [30], which consists of extending the best move from the previous search first, which accelerates the new search. Some variants perform a search with unbounded depth (that is, the depth of their search is not fixed) [31, 32, 33]. Unlike minimax with or without alpha-beta pruning, the exploration of these algorithms is the *best-first minimax search* [34]. To avoid

any confusion with some best-first approaches at fixed depth, we call this algorithm *Unbound Best-First Minimax*, or more succinctly UBFM. UBFM iteratively extends the game tree by adding the children of one of the leaves of the game tree having the same value as that of the root (minimax value). These leaves are the states obtained after having played one of the best sequences of possible actions given the current partial knowledge of the game tree. Thus, this algorithm iteratively extends the *a priori* best sequences of actions. These best sequences usually change at each extension. Thus, the game tree is non-uniformly explored by focusing on the *a priori* most interesting actions without exploring just one sequence of actions. In this article, we use the any-time version of UBFM [34], i.e. we leave a fixed search time for UBFM to decide the action to play. We also use transposition tables [35, 27] with UBFM, which makes it possible not to explicitly build the game tree and to merge the nodes corresponding to the same state. Algorithm 1 is the used implementation of UBFM in this paper<sup>4</sup>.

#### 2.2. Learning of Evaluation Functions

Reinforcement learning of evaluation functions can be done by different techniques [9, 2, 20, 36]. The general idea of reinforcement learning of state evaluation functions is to use a game tree search algorithm and an adaptive evaluation function  $f_{\theta}$ , of parameter  $\theta$ , (for example a neural network) to play a sequence of games (for example against oneself, which is the case in this article). Each game will generate pairs (s, v) where s is a state and v the value of s calculated by the chosen search algorithm using the evaluation function  $f_{\theta}$ . The states generated during one game can be the states of the sequence of states of the game [10, 37]. For example, in the case of *root bootstrapping* (technique that we call *root learning* in this article), the set of pairs used during the learning phase is  $D = \{(s, v) \mid s \in \mathbf{R}\}$  with **R** the set of states of the sequence of the game. In the case of the *tree bootstrapping* (tree learning) technique [37], the gen-

<sup>&</sup>lt;sup>4</sup>This implementation is a slight variant of Korf and Chickering algorithm. Their algorithm is very slightly more efficient but it offers less freedom: our algorithm behaves slightly differently depending on how we decide between two states having the same value. The exploration of states is identical between their algorithm and ours when with our algorithm equality is decided in deepest first. Our variant has been discovered independently of Korf and Chickering work.

Symbols	Definition
actions(s)	action set of the state s for the current player
$\operatorname{first\_player}(s)$	true if the current player of the state $s$ is the first player
terminal (s)	true if s is an end-game state
a(s)	state obtained after playing the action $a$ in the state $s$
time()	current time in seconds
S	keys of the transposition table $T$
Т	transposition table (contains functions as $v$ and/or $v'$ ; depends on the used search algorithm)
au	search time per action
t	time elapsed since the start of the reinforcement learning process
$t_{ m max}$	chosen total duration of the learning process
n(s,a)	number of times the action $a$ is selected in state $s$ (initially, $n(s,a)=0$ for all $s$ and $a$
v(s)	value of state s in the game tree according to the last tree search
v'(s,a)	value obtained after playing action $a$ in state $s$
c(s)	completion value of state $s$ (0 by default)
r(s)	resolution value of state $s$ (0 by default)
f(s)	the used evaluation function (first player point of view)
$f_{ heta}(s)$	adaptive evaluation function (of non-terminal game tree leaves ; first player point of view)
$f_{ m t}(s)$	evaluation of terminal states, e.g. gain game (first player point of view)
Gain function $b_t(s)$	0 if s is a draw, 1 if s is winning for the first player, $-1$ if s is losing for the first player
	a seach algorithm (it extends the game tree from $s$ , by adding new states in $S$
$\operatorname{search}(s, S, T, f_{\theta}, f_{t})$	and labeling its states, in particular, by a value $v(s)$ , stored in $T$ ,
	using $f_{\theta}$ as evaluation of the non-terminal leaves and $f_{\rm t}$ as evaluation of terminal states
action_selection( $s, S, T$ )	decides the action to play in the state $s$ depending on the partial game tree, i.e. on $S$ and $T$
processing(D)	various optional data processing: data augmentation (symmetry), experience replay,
update $(f_{\theta}, D)$	updates the parameter $\theta$ of $f_{\theta}$ in order for $f_{\theta}(s)$ is closer to $v$ for each $(s, v) \in D$

Table 1: Index of symbols

Algorithm 1 UBFM (Unbounded Best-First Minimax) algorithm : it computes the best action to play in the generated non-uniform partial game tree (see Table 1 for the definitions of symbols ; at any time  $T = \{v'(s, a) | s \in S, a \in \operatorname{actions}(s)\}$ ).

**Function** UBFM\_iteration (s, S, T)

```
if terminal (s) then
   return f(s)
  else
        \text{if} \ s \notin S \ \text{then} \\
           S \leftarrow S \cup \{s\}
           foreach a \in actions(s) do
           v'(s,a) \leftarrow f(a(s))
       else
           a_b \leftarrow \texttt{best\_action}(s)
           v'(s, a_b) \leftarrow \text{UBFM\_iteration}(a_b(s), S, T)
       a_b \leftarrow \texttt{best\_action}(s)
       return v'(s, a_b)
Function best_action (s, S, T)
  if first_player(s) then
       return \underset{a \in actions(s)}{\operatorname{arg\,max}} v'(s,a)
   else
       return \arg \min v'(s, a)
               a \in actions(s)
Function UBFM (s, \tau)
  t = \text{time}() while time() -t < \tau do UBFM_iteration(s, S, T)
  return best_action (s, S, T)
```

erated states are the states of the game tree built to decide which actions to play (which includes the states of the sequence of states of the game):  $D = \{(s, v) \mid s \in \mathbf{T}\}$  with  $\mathbf{T}$  the set of states of the partial game tree of the game. Thus, contrary to root bootstrapping, tree bootstrapping does not discard most of the information used to decide the actions to play. The values of the generated states can be their minimax values in the partial game tree built to decide which actions to play [37, 10]. Work on tree bootstrapping has been limited to reinforcement learning of linear functions of state features. It has not been formulated or studied in the context of reinforcement learning without knowledge and based on non-linear functions. Note that, in the case of AlphaGo Zero, the value of each generated state, the states of the sequence of the game, is the value of the terminal state of the game [2]. We call this technique terminal learning.

Generally between two consecutive games (between *match phases*), a *learning phase* occurs, using the pairs of the last game. Each learning phase consists in modifying  $f_{\theta}$  so that for all pairs  $(s, v) \in D$ ,  $f_{\theta}(s)$  sufficiently approaches v to constitute a good approximation. Note that, in the context of a variant, learning phases can use the pairs of *several* games. This technique is called *experience replay* [38]. Note that, adaptive evaluation functions  $f_{\theta}$  only serve to evaluate non-terminal states since we know the true value of terminal states.

*Remark* 1. There is another reinforcement learning technique for games: the Temporal Differences differences  $TD(\lambda)$  [10]. We can see the temporal differences as a kind of interpolation between root learning of a depth 1 minimax and the terminal learning. A variant has been proposed [39], called  $TD_{Leaf}(\lambda)$ , where the temporal difference technique is applied not to learn the value of the root but the value of the leaf of the principal variation of a minimax search (at any depth). A comparison between these techniques was made [37]. Finally, [40] describe a method for automatically tuning search-extension parameters, to decide which branches of the game tree must be explored during the search.

#### 2.3. Action Selection Distribution

One of the problems related to reinforcement learning is the *exploration-exploitation dilemma* [9]. It consists of choosing between exploring new states to learn new knowl-

edge and exploiting the acquired knowledge. Many techniques have been proposed to deal with this dilemma [41]. However, most of these techniques do not scale because their application requires memorizing all the encountered states. For this reason, in the context of games with large numbers of states, some approaches use probabilistic exploration [36, 2, 9, 42]. With this approach, to exploit is to play the best action and to explore is to play uniformly at random. More precisely, a parametric probability distribution is used to associate with each action its probability of being played. The parameter associated with the distribution corresponds to the exploration rate (between 0 and 1), which we denote  $\epsilon$  (the exploitation rate is therefore  $1 - \epsilon$ , which we denote  $\epsilon'$ ). The rate is often experimentally fixed. *Simulated annealing* [43] can, however, be applied to avoid choosing a value for this parameter. In this case, at the beginning of reinforcement learning, the parameter is 1 (we are just exploring). It gradually decreases until reaching 0 at the end of learning. The simplest action selection distribution is  $\epsilon$ -greedy [36] (of parameter  $\epsilon$ ). With this distribution, the action is chosen uniformly with probability  $\epsilon$  and the best action is chosen with probability  $1 - \epsilon$  (see also Algorithm 2).

Algorithm 2  $\epsilon$ -greedy algorithm with simulated annealing used in the experiments of this article (see Table 1 for the definitions of symbols).

**Function**  $\epsilon$ -greedy (s, v')

```
if probability \frac{t}{t_{\max}} thenif first_player(s) then| return \arg \max_{a \in \operatorname{actions}(s)} v'(s, a)else| return \arg \min_{a \in \operatorname{actions}(s)} v'(s, a)else| return a \in \operatorname{actions}(s) uniformly chosen.
```

The  $\epsilon$ -greedy distribution has the disadvantage of not differentiating the actions (except the best action) in terms of probabilities. Another distribution is often used, correcting this disadvantage. This is the *softmax* distribution [42, 9]. It is defined by  $P(a_i) = \frac{e^{v'(s,a_i)/\tau}}{\sum_{j=1}^{n} e^{v'(s,a_j)/\tau}}$  with n the number of children of the current state s,  $P(a_i)$  the probability of playing the action  $a_i$ ,  $v'(s, a_i)$  the value of the state obtained after

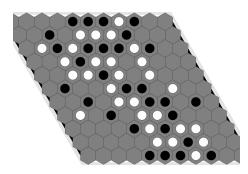


Figure 1: A Hex end game of size 11 (white wins)

playing  $a_i$  in  $s, i \in \{0, ..., n-1\}$ , and  $\tau \in ]0, +\infty[$  a parameter called temperature  $(\tau \simeq 0 : \text{exploitation}, \tau \simeq +\infty : \text{exploration}).$ 

## 2.4. Game of Hex

The game of Hex [44] is a two-player combinatorial strategy game. It is played on an empty  $n \times n$  hexagonal board. We say that a  $n \times n$  board is of size n. The board can be of any size, although the classic sizes are 11, 13 and 19. In turn, each player places a stone of his color on an empty cell (each stone is identical). The goal of the game is to be the first to connect the two opposite sides of the board corresponding to its color. Figure 1 illustrates an end game. Although these rules are simplistic, Hex tactics and strategies are complex. The number of states and the number of actions per state are very large, similar to the game of Go. From the board size 11, the number of states is, for example, higher than that of chess (Table 6 of [45]). For any board size, the first player has a winning strategy [46] which is unknown, except for board sizes smaller than or equal to 10 [47] (the game is weaky solved up to the size 10). In fact, resolving a particular state is PSPACE-complete [48, 49]. There is a variant of Hex using a swap rule. With this variant, the second player can play in first action a special action, called swap, which swaps the color of the two players (i.e. they swap their pieces and their sides). This rule reduces the imbalance between the two players (without the swap rule, the first player has a very strong advantage). It is generally used in competitions.

## 2.5. Hex Programs

Many Hex player programs have been developed. For example, Mohex 1.0 [50] is a program based on Monte Carlo tree search. It also uses many techniques dedicated to Hex, based on specific theoretical results. In particular, it is able to quickly determine a winning strategy for some states (without expanding the search tree) and to prune at each state many actions that it knows to be *inferior*. It also uses ad hoc knowledge to bias simulations of Monte Carlo tree search.

Mohex 2.0 [50] is an improvement of Mohex 1.0 that uses learned knowledge through supervised learning (namely correlations between victory and board patterns) to guide both tree exploration and simulations.

Other work then focused on predicting best actions, through supervised learning of a database of games, using a neural network [51, 52, 53]. The neural network is used to learn a *policy*, i.e. a prior probability distribution on the actions to play. These prior probabilities are used to guide the exploration of Monte Carlo tree search. First, there is Mohex-CNN [15] which is an improvement of Mohex 2.0 using a convolutional neural network [54]. A new version of Mohex was then proposed: Mohex-3HNN [16]. Unlike Mohex-CNN, it is based on a residual neural network [55]. It calculates, in addition to the policy, a value for states *and* actions. The value of states replaces the evaluation of states based on simulations of Monte Carlo tree search. Adding a value to actions allows Mohex-HNN to reduce the number of calls of the neural network, improving performance. Mohex-3HNN is the best Hex program. It wons Hex size 11 and 13 tournaments at 2018 Computer Olympiad [24].

Programs which learn the evaluation function by reinforcement have also been designed. These programs are NeuroHex [36], EZO-CNN [56], DeepEzo [57] and ExIt [20]. They learn from self-play. Unlike the other three programs, NeuroHex performs supervised learning (of a common Hex heuristic) followed by reinforcement learning. NeuroHex also starts its games with a state from a database of games. EZO-CNN and DeepEzo use knowledge to learn winning strategies in some states. DeepEzo also uses knowledge during confrontations. ExIt learns a policy in addition to the value of states and it is based on MCTS. It is the only program to have learned to play Hex without using knowledge. This result is, however, limited to the board size 9. A comparison of

Programs	Size	Search	Learning Network		Use	
Mohex-CNN	13	MCTS	supervised	convolutional	policy	
Mohex-3HNN	13	MCTS	supervised	residual	policy, state, action	
NeuroHex	13	none	supervised, reinforcement	convolutional	state	
EZO-CNN	7, 9, 11	Minimax	reinforcement convolutio		state	
DeepEZO	13	Minimax	reinforcement	convolutional	policy, state	
ExIt	9	MCTS	reinforcement	convolutional	policy, state	

Table 2: Comparison of the main features of the latest Hex programs. These characteristics are respectively the board sizes on which learning is based, the used tree search algorithm, the type of learning, the type of neural network and its use (to approximate the values of states, actions and/or policy.

the main characteristics of these different programs is presented in Table 2.

## 2.6. Games of Paper Experiments

We now briefly present the other games on which experiments are performed in this article, namely: Surakarta, Othello, Outer Open Gomoku, Clobber, Breakthrough, Amazons, Lines of Action, Santorini. They are all board games. All of these games (except Santorini so far) are present and recurring at the Computer Olympiads, the worldwide multi-games event in which computer programs compete against each other. Moreover, all these games (and their rules) are included (and available for free) in Ludii [58], a general game system.

# 2.6.1. Surakarta

Surakarta is a move and capture game (like the checkers game), the object of the game being to take all the opposing pieces. In his turn, a player can either move a piece to an empty square at a distance of 1 or move a piece to a square occupied by an opponent's piece under certain conditions and according to a mechanism specific to Surakarta (based on a movement circuit dedicated only to capture), allowing "long distance" capture.

# 2.6.2. Othello

Othello (also called Reversi) is a linear territory and encirclement game whose goal is to have more pieces than your opponent. In his turn, a player places a piece of his color on the board (only if he can make an encirclement, otherwise he pass his turn). There is an encirclement if an opponent's line of pieces has at its two ends the piece that has just been placed and another piece of the player performing the encirclement. As a result of this encirclement, the encircled pieces are replaced by pieces from that player.

## 2.6.3. Outer Open Gomoku

Outer Open Gomoku is an alignment game. The object of the game is to line up at least 5 pieces of its color. On his turn, a player places a piece of his color. In the first turn, the first player can only place a piece at a distance of 2 from the sides of the board.

## 2.6.4. Clobber

Clobber is a move and capture game. The goal is to be the last player to have played. A player can play if he can orthogonally move one of his pieces onto a neighboring square on which there is an opponent's piece. This movement is always a capture (the opponent's piece is removed from the board).

## 2.6.5. Breakthrough

Breakthrough is a move and capture game. The object of the game is to be the first to make one of his pieces reach the other side of the board. A piece can only move by moving forward one square (straight or diagonal). A capture can only be done diagonally.

# 2.6.6. Amazons

Amazons is a move and blocking game. In turn, a player moves one of his pieces in a straight line in any direction (like the queen of the game of Chess). Then he places a neutral piece (blocking movements like the players' pieces) in any direction starting from the new position of the piece just moved (in the manner of the queen of the game of Chess). The goal of the game is to be the last to play.

## 2.6.7. Lines of Action

Lines of Action is a game of movement and regrouping. On his turn, a player can move one of his pieces in one direction as many squares as there are pieces in that direction. A piece cannot move if there is an opponent's piece in its path, unless it is the square to arrive (in which case a capture is made). The goal is to have all of its pieces connected (at the same time).

## 2.6.8. Santorini

Santorini is a three-dimensional building and moving game. The goal of the game is to reach the 3rd floor of a building. In his turn, a player moves one of his pieces by one square then places the first floor on an adjacent empty square or increases a pre-existing construction by one floor (on which no player's piece is located). A piece cannot move to a square having strictly more than one floor more than the square where it is located (a piece only go up one floor at a time and can descend as many floors as wanted). A move cannot be made to a square with 4 floors. A construction cannot be done on a square of 4 floors. A player who cannot play loses. Advanced mode (i.e. the use of power cards) is not used in the experiments in this article.

## 3. Data Use in Game Learning

In this section, we adapt and study tree learning (see Section 2.2) in the context of reinforcement learning and the use of non-linear adaptive evaluation functions. For this, we compare it to root learning and terminal learning in this context. We start by adapting tree learning, root learning, and terminal learning. Next, we describe the experiment protocol common to several sections of this article. Finally, we expose the comparison of tree learning with root learning and terminal learning.

#### 3.1. Tree Learning

As we saw in Section 2.2, tree learning consists in learning the value of the states of the partial game tree obtained at the end of the game. Root learning consists in learning the values of the states of the sequence of states of the game (the value of each state is its value in the search tree). Terminal learning consists in learning the values of the sequence of states of the game but the value of each state is the value of the terminal state of the game (i.e. the gain of the game). Data to learn after each game, can be modified by some optional data processing methods, such as experience replay (see Section 2.2). The learning phase uses a particular update method so that the adaptive evaluation function fit the chosen data. The adaptation of tree learning, root learning, and terminal learning are given respectively in Algorithm 3, Algorithm 4, and Algorithm 5. In this article, we use experience replay as data processing method (see Algorithm 6; its parameter are the memory size  $\mu$  and the sampling rate  $\sigma$ ). In addition, we use a stochastic gradient descent as update method (see Algorithm 7; its parameter is B the batch size). Formally, in Algorithm 3, Algorithm 4, and Algorithm 5, we have: processing(D) is experience\_replay(D,  $\mu$ ,  $\sigma$ ) and update( $f_{\theta}$ , D) is stochastic\_gradient\_descent( $f_{\theta}$ , D, B). Finally, we use  $\epsilon$ -greedy as default action selection method (i.e. action\_selection(s, S, T) is  $\epsilon$ -greedy(s, T.v') (T stores the children value function v'; see Algorithm 2)).

## 3.2. Common Experiment Protocol

The experiments of several sections share the same protocol. It is presented in this section. The protocol is used to compare different variants of reinforcement learning algorithms. A variant corresponds to a certain combination of elementary algorithms. More specifically, a combination consists of the association of a search algorithm (iterative deepening alpha-beta (with move ordering), MCTS (UCT with c = 0.4 as exploration constant), UBFM, ...), of an action selection method ( $\epsilon$ -greedy distribution (used by default), softmax distribution, ...), a terminal evaluation function  $f_t$  (the classic game gain (used by default), ...), and a procedure for selecting the data to be learned (root learning, tree learning, or terminal learning). The protocol consists in carrying out a reinforcement learning of 48 hours for each variant. At several stages of the learning process, matches are performed using the adaptive evaluation functions obtained by the different variants. Each variant is then characterized by a winning percentage at each stage of the reinforcement learning process. More formally, we denote by  $f_{\theta_h}^c$  the eval-

uation generated by the combination c at the hour h. Each combination is evaluated every hour by a winning percentage. The winning percentage of a combination c at a hour  $h \leq 48$  (i.e. of  $f_{\theta_h}^c$ ) is computed from matches against each combination c' at final time h = 48, i.e. against each  $f_{\theta_{48}}^{c'}$  (there is one match in first player and another in second player per pair of combination). The matches are made by using alpha-beta at depth 1.

This protocol is repeated several times for each experiment in order to reduce the statistical noise in the winning percentages obtained for each variant (the obtained percentage is the average of the percentages of repetitions). The winning percentages are then represented in a graph showing the evolution of the winning percentages during training.

In addition to the curve, the different variants are also compared in relation to their final winning percentage, i.e. at the end of the learning process. Unlike the experiment of the evolution of winning percentages, in the comparison of the different variants

Algorithm 3 Tree learning (tree bootstrapping) algorithm (see Table 1 for the definitions of symbols). In this context, S is the set of states which are non-leaves or terminal.

```
Function tree_learning (t_{max})

t_0 \leftarrow time()

while time() - t_0 < t_{max} do

s \leftarrow initial_game_state()

S \leftarrow \emptyset

T \leftarrow \{\}

while \negterminal(s) do

S, T \leftarrow search(s, S, T, f_{\theta}, f_t)

a \leftarrow action_selection(s, S, T)

s \leftarrow a(s)

D \leftarrow \{(s, v(s)) \mid s \in S\}

D \leftarrow processing(D)

update(f_{\theta}, D)
```

at the final stage, each evaluation  $f_{\theta_{48}}^c$  confronts each other evaluation  $f_{\theta_{48}}^{c'}$  of all the repetitions. In other words, this experiment consists of performing an all-play-all tournament with all the evaluation functions generated during the different repetitions. The presented winning percentage of a combination is still the average over the repetitions. The matches are also made by using alpha-beta at depth 1. These percentages are shown in tables.

*Remark* 2. The used version of MCTS does not performed random simulations for evaluating the leaves. Instead, leaves are evaluated by a neural network. No policies are used (unless it is explicitly specified).

*Remark* 3. All the experiments involving MCTS were also performed with  $c = \sqrt{2}$  as exploration constant. The results are similar.

Algorithm 4 Root learning (root bootstrapping) algorithm (see Table 1 for the definitions of symbols).

```
Function root_learning(t_{\max}) | t_0 \leftarrow time()
```

```
 \begin{array}{c|c} \textbf{while time ()} - t_0 < t_{\max} \, \textbf{do} \\ s \leftarrow \texttt{initial\_game\_state ()} \\ S \leftarrow \emptyset \\ T \leftarrow \{\} \\ D \leftarrow \emptyset \\ \textbf{while } \neg \texttt{terminal } (s) \, \textbf{do} \\ & S, T \leftarrow \texttt{search } (s, S, T, f_\theta, f_\texttt{t}) \\ a \leftarrow \texttt{action\_selection } (s, S, T) \\ D \leftarrow D \cup \{(s, v(s))\} \\ s \leftarrow a(s) \\ D \leftarrow \texttt{processing } (D) \\ \texttt{update } (f_\theta, D) \\ \end{array}
```

# 3.2.1. Technical Details

The used parameters are: search time per action  $\tau = 2s$ , batch size B = 128, memory size  $\mu = 10^6$ , sampling rate  $\sigma = 4\%$  (see Section 3.1). Moreover, the used

Algorithm 5 Terminal learning algorithm (see Table 1 for the definitions of symbols).

```
Function terminal_learning (t<sub>max</sub>)

t_0 \leftarrow time()

while time() - t_0 < t<sub>max</sub> do

s ← initial_game_state()

S ← Ø

T ← {}

G ← {s}

while ¬terminal(s) do

S, T ← search(s, S, T, f_{\theta}, f_t)

a ← action_selection(s, S, T)

s ← a(s)

G ← G ∪ {s}

D ← {(s', f_t(s)) | s' ∈ G}

D ← processing(D)

update(f_{\theta}, D)
```

Algorithm 6 Experience replay (replay buffer) algorithm used in the experiments of this article.  $\mu$  is the memory size and  $\sigma$  is the sampling rate. M is the memory buffer (global variable initialized by an empty queue). If the number of data is less than  $\sigma \cdot \mu$ , then it returns all data (no sampling). Otherwise, it returns  $\sigma \cdot \mu$  random elements.

```
Function experience_replay (D, \mu, \sigma)
add the elements of D in M
if |M| > \mu then
| remove the oldest items of M to have |M| = \mu
if |M| \le \sigma \cdot \mu then
| return M
return a list of random items of M whose size is \sigma \cdot \mu
```

adaptive evaluation function for each combination is a convolutional neural network [54] having three convolution layers<sup>5</sup> followed by a fully connected hidden layer. For each convolutional layer, the kernel size is  $3 \times 3$  and the filter number is 64. The number of neurons in the fully connected layer is 100. The margin of each layer is zero. After each layer except the last one, the ReLU activation function [61] is used. The output layer contains a neuron. When the classical terminal evaluation is used, tanh is the output activation function. Otherwise, there is no activation function for the output.

*Remark* 4. In this paper, filter numbers and numbers of neurons are chosen in order to there are about the same number of variables in the convolution layers and in the dense layers.

#### 3.3. Comparison of Learning Data Selection Algorithms

We now compare tree learning, root learning and terminal learning, using the protocol of Section 3.2. Each combination uses either tree learning or root learning or terminal learning. Moreover, each combination uses either iterative deepening alphabeta (denoted by ID) or MCTS. Furthermore, each combination uses  $\epsilon$ -greedy as action selection method (see Section 3.1) and the classical terminal evaluation (1 if the first player wins, -1 if the first player loses, 0 in case of a draw). There are a total of 6

Algorithm 7 Stochastic gradient descent algorithm used in the experiments of this article. It is based on Adam optimization (1 epoch per update) [59] and  $L_2$  regularization (with  $\lambda = 0.001$  as parameter) [60] and implemented with tensorflow. *B* is the batch size (see Table 1 for the definitions of the other symbols)

Function stochastic\_gradient\_descent  $(f_{\theta}, D, B)$ Split D in m disjoint sets, denoted  $\{D_i\}_{i=1}^m$ , such that  $D = \bigcup_{i=1}^m D_i$  and  $|D_i| = B$ for each  $i \in \{1, \dots, m-1\}$ for each  $i \in \{1, \dots, m\}$  do minimize  $\sum_{(s,v)\in D_i} (f_{\theta}(s) - v)^2$  by using Adam and  $L_2$  regularization

<sup>&</sup>lt;sup>5</sup>There is an exception: for the game Surkarta, there is only two convolution layers.

combinations. The experiment was repeated 32 times. The winning percentage of a combination for each game and for each evaluation step (i.e. each hour) is therefore calculated from 192 matches. The winning percentage curves are shown in Figure 2. The final winning percentages are shown in Table 3. Each percentage of the table has required 6144 matches. ID is first on all games except in Outer Open Gomoku where it is second (MCTS root learning is first) and in Surakarta (MCTS with tree learning is first). MCTS with root learning is better than MCTS with tree learning except in Break-through and Surakarta. At Hex and Amazons, MCTS with root learning gives better results throughout the learning process but ends up being caught up by ID with terminal learning. Terminal learning performs worse everywhere, except in a few cases where it is very slightly better. On average, ID with tree learning is better (71% win), then MCTS with root learning is second (9% lower win percentage), followed by MCTS with tree learning (18% lower to ID).

In conclusion, tree learning with ID performs much better than other combinations, although the results are very tight at Amazons, Hex, and Outer Gomoku with MCTS with root learning.

## 4. Tree Search Algorithms for Game Learning

In this section, we introduce a new tree search algorithm, that we call *descent minimax* or more succinctly *descent*, dedicated to be used during the learning process. It requires tree learning (combining it with root learning or terminal learning is of no interest<sup>6</sup>). After presenting descent, we compare it to MCTS with root learning and with tree learning, to iterative deepening alpha-beta with root learning and with tree learning and to UBFM with tree learning.

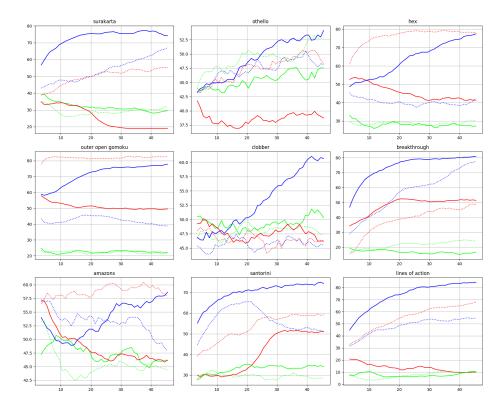


Figure 2: Evolutions of the winning percentages of the combinations of the experiment of Section 3.3, i.e. MCTS (dotted line) or iterative deepening alpha-beta (continuous line) with tree learning (blue line) or root learning (red line) or terminal learning (green line). The display uses a simple moving average of 6 data.

	tree learning		root learning		terminal learning	
	MCTS	ID	MCTS	ID	MCTS	ID
Othello	48.8%	55.4%	51.0%	31.9%	52.9%	43.7%
Hex	45.1%	79.8%	79.8%	44.1%	29.8%	27.8%
Clobber	41.5%	62.5%	50.0%	45.5%	45.7%	49.8%
Outer Open Gomoku	40.3%	80.0%	87.3%	48.8%	21.6%	23.1%
Amazons	46.2%	58.8%	56.1%	44.5%	39.4%	46.0%
Breakthrough	78.1%	79.2%	45.5%	50.6%	22.3%	14.3%
Santorini	50.2%	73.8%	60.5%	51.3%	29.4%	36.5%
Surakarta	69.4%	65.2%	56.2%	20.8%	28.9%	23.6%
Lines of Action	58.8%	81.7%	67.7%	9.6%	6.9%	2.7%
mean	53.1%	70.7%	61.6%	38.5%	30.8%	29.7%

Table 3: Final winning percentages of the combinations of the experiment of Section 3.3 (ID: iterative deepening alpha-beta). Reminder: the percentage is the average over the repetitions, of the winning percentage of a combination against each other combination, in first and second player (see 3.2; 95% confidence intervals: max  $\pm 0.85\%$ ).

Algorithm 8 Descent tree search algorithm (see Table 1 for the definitions of symbols ; note: S is the set of states which are non-leaves or terminal and T = (v, v')).

```
Function descent_iteration (s, S, T, f_{\theta}, f_{t})
     if terminal (s) then
           S \leftarrow S \cup \{s\}
         v(s) \leftarrow f_{t}(s)
     else
           if s \notin S then
                S \leftarrow S \cup \{s\}
                 foreach a \in actions(s) do
                      if terminal (a(s)) then
                            S \leftarrow S \cup \{a(s)\}
                            v'(s,a) \leftarrow f_{t}(a(s))
                            v(a(s)) \leftarrow v'(s,a)
                      else
           \begin{vmatrix} & | & | & v'(s,a) \leftarrow f_{\theta}\left(a(s)\right)\\ a_b \leftarrow \texttt{best\_action}\left(s\right) \end{vmatrix}
           v'(s, a_b) \leftarrow \texttt{descent\_iteration}(a_b(s))
           a_b \leftarrow \texttt{best\_action}(s)
           v(s) \leftarrow v'(s, a_b)
     return v(s)
```

Function best\_action (s)

 $\begin{array}{c|c} \text{if first_player}(s) \text{ then} \\ | & \text{return } \underset{a \in \text{actions}(s)}{\arg \max} v'(s,a) \\ \text{else} \\ | & \text{return } \underset{a \in \text{actions}(s)}{\arg \min} v'(s,a) \end{array}$ 

Function descent (s, S, T,  $f_{ heta}, f_{ ext{t}}, au$ )

t= time()  $\label{eq:time}$  while time()  $-t<\tau$  do descent\_iteration(s, S, T,  $f_{\theta},\,f_{\rm t})$  return  $S,\,T$ 

## 4.1. Descent: Generate Better Data

Thus, we present *descent*. It is a modification of UBFM which builds a different, deeper, game tree, to be combined with tree learning. The idea of descent is to combine UBFM with deterministic end-game simulations providing interesting values from the point of view of learning. The algorithm descent (Algorithm 8) recursively selects the best child of the current node, which becomes the new current node. It adds the children of the current node if they are not in the tree. It performs this recursion from the root (the current state of the game) until reaching a terminal node (an end game). It then updates the value of the selected nodes (minimax value). The algorithm descent repeats this recursive operation starting from the root as long as there is some search time left. Descent is almost identical to UBFM. The only difference is that descent performs an iteration until reaching a terminal state while UBFM performs this iteration until reaching a leaf of the tree (UBFM stops the iteration much earlier). In other words, during an iteration, UBFM just extends one of the leaves of the game tree while descent recursively extends the best child from this leaf until reaching the end of the game. The algorithm *descent* has the advantage of UBFM, i.e. to perform a longer search to determine a better action to play. By learning the values of the game tree (by using for example tree learning), it also has the advantage of a minimax search at depth 1, i.e. to raise the values of the terminal nodes to the other nodes more quickly. In addition, the states thus generated are closer to the terminal states. Their values are therefore better approximations.

*Remark* 5. In the experiments of this article, when there is a value tie in best\_action(s), the tie is broken at random. An alternative could be to choose the subtree whose principal variation is the least deep.

<sup>&</sup>lt;sup>6</sup>Because, with root and terminal learning, the extra computations of descent compared to UBFM with the same number of iterations, do not change, in general, the values to learn and the move to play (but the time to perform the iterations is much longer).

## 4.2. Comparison of Search Algorithms for Game Learning

We now compare descent with tree learning to MCTS with root learning and with tree learning, to iterative deepening alpha-beta with root learning and with tree learning, and to UBFM with tree learning, using the protocol of Section 3.2. Each combination uses one of these tree search algorithms combined with tree/root learning. There are a total of 6 combinations. The experiment was repeated 32 times. The winning percentage of a combination for each game and for each evaluation step (i.e. each hour) is therefore calculated from 192 matches. The winning percentage curves are shown in Figure 3. The final winning percentages are shown in Table 4. Each percentage of the table has required 6144 matches. It is descent which gets the best curves on all games. For two games (Surakarta and Outer Open Gomoku), the difference with UBFM is very narrow but the results remain better than the classic approaches (MCTS and alpha-beta). On each game, descent obtains a final percentage higher than all the other combinations (except in Santorini where it is 2% lower than UBFM, the best algorithm at this game). On average over all games, descent has 82% win and is above UBFM, the second best combination, by 18% and ID with tree learning, the third best combination, by 34%. In conclusion, descent (with tree learning) is undoubtedly the best combination. UBFM (with tree learning) is the second best combination, sometimes very close to descent performances and sometimes very far, but always superior to other combinations (slightly or largely depending on the game).

## 5. Reinforcement Heuristic to Improve Learning Performance

In this section, we propose the technique of *reinforcement heuristic*, which consists to replace the classical terminal evaluation function – that we denote by  $b_t$ , which returns 1 if the first player wins, -1 if the second player wins, and 0 in case of a draw [36, 2, 16] – by another heuristic to evaluate terminal states during the learning process. By using this technique, non-terminal states are therefore evaluated differently, partial game trees and thus matches during the learning process are different, which can impact the learning performances. We start by defining what we call a reinforcement heuristic and we offering several reinforcement heuristics. Then, we propose a complementary

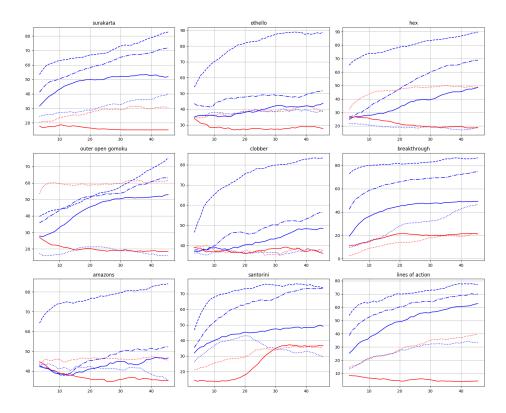


Figure 3: Evolutions of the winning percentages of the combinations of the experiment of Section 4.2, i.e. of descent (dashed line), UBFM (dotted dashed line), MCTS (dotted line), iterative deepening alpha-beta (continuous line) with tree learning (blue line) or root learning (red line). The display uses a simple moving average of 6 data.

		tree lea	root learning			
	descent	UBFM	MCTS	ID	MCTS	ID
Othello	89.4%	47.2%	37.7%	42.7%	44.9%	22.1%
Hex	94.9%	71.7%	20.5%	50.7%	50.2%	20.5%
Clobber	83.0%	56.7%	32.9%	48.9%	42.0%	35.5%
Outer Open Gomoku	77.6%	63.9%	18.0%	51.6%	64.7%	18.2%
Amazons	84.3%	55.3%	32.5%	46.3%	43.7%	31.7%
Breakthrough	86.5%	72.5%	45.5%	47.6%	19.2%	21.0%
Santorini	69.9%	71.8%	31.9%	47.6%	37.7%	40.2%
Surakarta	82.8%	69.7%	41.2%	42.1%	29.4%	14.5%
Lines of Action	73.4%	66.9%	36.1%	57.4%	39.4%	3.7%
mean	82.4%	64.0%	32.9%	48.3%	41.2%	23.1%

Table 4: Final winning percentages of the combinations of the experiment of Section 4.2 (ID: iterative deepening alpha-beta; see 3.2; 95% confidence intervals: max  $\pm 0.85\%$ )

technique, that we call *completion*, which corrects state evaluation functions taking into account the resolution of states. Finally, we compare the reinforcement heuristics that we propose to the classical terminal evaluation function.

#### 5.1. Some Reinforcement Heuristics

A reinforcement heuristic is a terminal evaluation function that is more expressive than the classical terminal function, i.e. the game gain.

**Definition 6.** Let  $b_t$  the game gain function of a game (i.e.  $b_t$  returns 1 if the first player wins, -1 if the second player wins, and 0 in case of a draw).

A reinforcement heuristic  $h_r$  is a function that preserves the order of the game gain function: for any two terminal states of the game  $s, s', b_t(s) < b_t(s')$  implies  $h_r(s) < h_r(s')$ .

In the following subsections, we propose different reinforcement heuristics.

# 5.1.1. Scoring

Some games have a natural reinforcement heuristic: the game score. For example, in the case of the game Othello (and in the case of the game Surakarta), the game score is the number of its pieces minus the number of pieces of his opponent (the goal of the game is to have more pieces than its opponent at the end of the game). The scoring heuristic used as a reinforcement heuristic consists of evaluating the terminal states by the final score of the game. With this reinforcement heuristic, the adaptive evaluation function will seek to learn the score of states. In the context of an algorithm based on minimax, the score of a non-terminal state is the minimax value of the subtree starting from this state whose terminal leaves are evaluated by their scores. After training, the adaptive evaluation function then contains more information than just an approximation of the result of the game, it contains an approximation of the score of the game. If the game score is *intuitive*, this should improve learning performances.

*Remark* 7. In the context of the game of the Amazons, the score is the size of the territory of the winning player, i.e. the squares which can be reached by a piece of the winning player. This is approximately the number of empty squares.

## 5.1.2. Additive and Multiplicative Depth Heuristics

Now we offer the following reinforcement heuristic: the *depth heuristic*. It consists in giving a better value to the winning states close to the start of the game than to the winning states far from the start. Reinforcement learning with the depth heuristic, it is learning the duration of matches in addition to their results. This learned information is then used to try to win as quickly as possible and try to lose as late as possible. The hypothesis of this heuristic is that a state close to the end of the game has a more precise value than a state more distant and that the duration of the game is easily learned. Under this assumption, with this heuristic, we will take less risk to try to win as quickly as possible and to lose as late as possible. In addition, with a long game, a player in difficulty will have more opportunities to regain the upper hand. We propose two realizations of the depth heuristic: the *additive depth heuristic*, that we denote by  $p_t$ , and the *multiplicative depth heuristic*, that we denote by  $p_t'$ . The evaluation function  $p_t$  returns the value l if the first player wins, the value -l if the second player wins, and 0 in case of a draw, with l = P - p + 1 where P is the maximum number of playable actions in a game and p is the number of actions played since the beginning of the game. For the game of Hex, l is the number of empty cells on the board plus 1. For the games where P is very large or difficult to compute, we can instead use  $l = \max\left(1, \tilde{P} - p\right)$  with  $\tilde{P}$  a constant approximating P (close to the empirical average length of matches). The evaluation function  $p_t'$  is identical except that l satisfies  $l = \frac{\tilde{P}}{p}$ . *Remark* 8. Note that the idea of fast victory and slow defeat has already been proposed but not used in a learning process [62].

#### 5.1.3. Cummulative Mobility

The next reinforcement heuristic that we propose is *cumulative mobility*. It consists in favoring the games where the player has more possibility of action and where his opponent has less. The implementation used in this article is as following. The value of a terminal state is  $\frac{M_1}{M_2}$  if the first player wins,  $-\frac{M_2}{M_1}$  if the second player wins, and 0 in case of a draw, where  $M_1$  is the mean of the number of available actions in each turn of the first player since the start of the game and  $M_2$  is the mean of the number of available actions in each turn of the second player since the start of the game.

#### 5.1.4. Piece Counting: Presence

Finally, we propose as reinforcement heuristic: the *presence* heuristic. It consists in taking into account the number of pieces of each player and starts from the assumption that the more a player has pieces the more this one has an advantage. There are several implementations for this heuristic, we use in this article the following implementation: the heuristic value is  $\max(n_1 - n_2, 1)$  if the first player wins,  $\min(n_1 - n_2, -1)$  if the second player wins, and 0 in case of a draw, where  $n_1$  is the number of pieces of the first player and  $n_2$  is the number of pieces of the second player. Note that in the games Surakarta and Othello, the score corresponds to a presence heuristic.

## 5.2. Completion

Relying solely on the value of states calculated from the terminal evaluation function *and* the adaptive evaluation function can sometimes lead to certain aberrant behaviors. More precisely, if we only seek to maximize the value of states, we will then

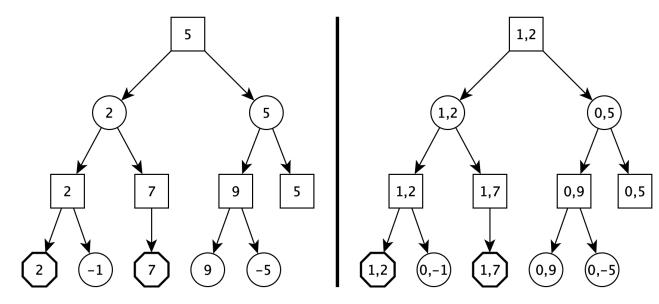


Figure 4: The left graph is a game tree where maximizing does not lead to the best decision; the right graph is the left game tree with completion (nodes are labeled by a pair of values) and thus maximizing leads to the best decision (square node: first player node (max node), circle node: second player node (min node), octagon: terminal node).

choose to play a state *s* rather than another state *s'* if *s* is of greater value than *s'* even if *s'* is a winning resolved state (a state is *resolved* if we know the result of the game starting from this state in which the two players play optimally). A search algorithm can resolve a state. This happens when all the leaves of the subtree starting from this state are terminal. Choosing *s* rather than *s'*, a winning resolved state, is an error<sup>7</sup> when *s* is not resolved (or when *s* is resolved and is not winning). By choosing *s*, guarantee of winning is lost. The left graph of Figure 4 illustrates such a scenario. It is therefore necessary to take into account both the value of states and the resolution of states. The completion technique, which we propose in this section, is one way of doing it. It consists, on the one hand, in associating with each state *s* a completion value c(s) and a resolution value r(s). The completion value c(s) of a leaf state *s* is 0 if the state *s* is not

<sup>&</sup>lt;sup>7</sup>There is perhaps, in certain circumstances, an interest in making this error from the point of view of learning.

terminal or if it is a draw, 1 if it is a winning terminal state and -1 if the state is a losing terminal state. The value c(s) of a non-leaf state s is computed as the minimax value of the subtree of the partial game tree starting from s where the leaves are evaluated by their completion value. The resolution value r(s) of a leaf state s is 0 if the state s is not terminal and 1 if it is terminal. The resolution value r(s) of a non-leaf state s is 1 if |c(s)| = 1. Otherwise, r(s) is the minimum of the resolution values of the childen of s. The completion technique consists, on the other hand, in using  $c(\cdot)$  to compute  $v(\cdot)$ . For this, states are compared from pairs  $(c(\cdot), v(\cdot))$ , by using the lexicographic order (instead of just compare states from  $v(\cdot)$ ). More precisely, the value v(s) of a state s is computed in calculating (c(s), v(s)) as the minimax value of the subtree of the partial game tree starting from s where the leaves l are evaluated by (c(l), v(l)). Thus, the value v(s) of a winning state s is always the value of the corresponding winning terminal leaf. Finally, with the completion technique, to decide which action to play during the search, we choose the best unresolved action if it exists and otherwise the best resolved action (i.e. we choose the action which maximize (-r(s), c(s), v(s))in a max state and which minimize (r(s), c(s), v(s)) in a min state). The right graph of Figure 4 illustrates the use of completion. The use of the resolution of states also makes it possible to stop the search in the resolved subtrees and thus to save computing time. Descent algorithm modified to use the completion and the resolution stop is described in Algorithm 9. With completion, descent always chooses an action leading to a winning resolved state and never chooses, if possible, an action leading to a losing resolved state.

*Remark* 9. For a game where there is no draw, the computation of the resolution value is not necessary (all necessary information is in the completion value).

*Remark* 10. A variant consists in performing a strong resolution, i.e. to compute  $r(\cdot)$  for non-leaf states in the following way: r(s) is (in all cases) the minimum of the resolution values of the childen of the state s. With this variant, unnecessary calculations are made to determine the best action to play. However, the additional values may be useful for learning. It is not clear which variant is the best.

Another variant is possible, it consists in not using the completition  $c(\cdot)$  to compute

 $v(\cdot)$ . With this version, v(s) is the minimax value of the subtree starting from s where the leaf nodes l are evaluated by v(l). With this variant, for certain resolved states, instead of learning a lower bound (in absolute value), which is an exact value, we learn an estimated value, which could be a potentially an overestimated value. It is not clear which variant is the best.

We also propose to use the resolution of states with action selections, to reduce the duration of games and therefore *a priori* the duration of the learning process: always play an action leading to a winning resolved state if it exists and never play an action leading to a losing resolved state if possible. Thus, if among the available actions we know that one of the actions is winning, we play it. If there is none, we play according to the chosen action selection method among the actions not leading to a resolved state (if possible). If it is not possible, we play the best action leading to a state resolved as a draw and if there is none, we play the best action leading to a losing resolved state. We call it *completed action selection*.

*Remark* 11. It is not clear, however, that this improves performance as it prunes a portion of the game tree in the exploration whose values could be useful for learning (but could also negatively impact the learning performance).

#### 5.3. Comparison of Reinforcement Heuristics

We now compare the different heuristics that we have proposed to the classical terminal evaluation function  $b_t$  on different games, using the protocol of Section 3.2. Each combination uses descent with completion (Algorithm 9) and completed  $\epsilon$ -greedy (see Algorithm 2 and Section 5.2). Each combination uses a different terminal evaluation function. These terminal evaluations are the classical ("binary") evaluation function  $b_t$ , the additive depth heuristic, the multiplicative depth heuristic, the scoring heuristic, the cummulative mobility, and the presence heuristic. Other parameters are the same as Section 3.3. There are, at most, a total of 6 combinations per game (on some games, some heuristics are not evaluated because they are trivially of no interest or equivalent to another heuristic). The experiment was repeated 48 times. The winning percentage of a combination for each game and for each evaluation step (i.e. each hour) is therefore calculated from 96 to 192 matches. The final winning percentages are shown in Table 5. Each percentage of the table has required between 4608 and 9216 matches. On average and in 7 of the 9 games, the classic terminal heuristic has the worst percentage (exception are Othello and Lines of Action). In scoring games, scoring is the best heuristic, as we might expect. Leaving aside the score heuristic, with the exception of Surakarta, Othello and Clobber, it is one of the two depth heuristics that has the best winning percentage. In Surakarta and Clobber, mobility is just ahead of the depth heuristics. On average, using the additive depth heuristic instead of using the classic evaluation increases the winning percentage by 15%, and using the best depth heuristic increases the winning percentage by 19%. The winning percentage curves are shown in Figure 5. The final percentages summarize the curves quite well. Note however, on the one hand, that the clear impact compared to the other heuristics (except score) of the additive depth heuristic to Breakthrough, Amazons, Hex, and Santorini and of the multiplicative depth heuristic to Clobber, Hex, and Open Outer Gomoku. In conclusion, the use of generic reinforcement heuristics has significantly improved performances and the depth heuristics are prime candidates as a powerful generic reinforcement heuristic.

#### 6. Search Algorithms for Game Playing

In this section, we propose another variant of UBFM, dedicated to be used in competition mode. Then, we compare it with other tree search algorithms.

## 6.1. Unbound Best-First Minimax with Safe Decision: $UBFM_s$

Thus, we propose a modification of UBFM, denoted UBFM<sub>s</sub>. It aims to provide a safer game. The action UBFM chooses to play is the one that leads to the state of best value. In some cases, the (*a priori*) best action can lead to a state that has not been sufficiently visited (such as a non-terminal leaf). Choosing this action is therefore a risky decision. We propose, to avoid this problem, a different decision that aims to play the safest action, in the same way as MCTS (max child selection [13]). If no action leads to a winning resolved state, the action chosen by UBFM<sub>s</sub> is the one that has been the most selected (since the current state of the game) during the exploration

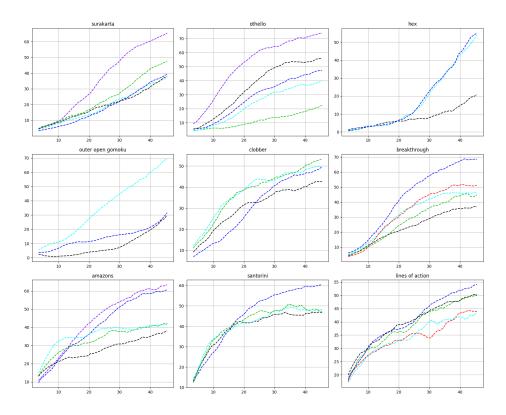


Figure 5: Evolutions of the winning percentages of the combinations of the experiment of Section 5.3, i.e. the use of the following heuristics: classic (black line), score (purple line), additive depth (blue line), multiplicative depth (turquoise line), cumulative mobility (green line), and presence (red line). The display uses a simple moving average of 6 data.

				depth		
	classic	score	additive	multiplicative	mobility	presence
Othello	49.8%	70.6%	50.1%	48.9%	18.5%	score
Hex	33.3%	Х	66.1%	60.4%	Х	Х
Clobber	43.7%	Х	47.0%	49.8%	53.5%	Х
Outer Open Gomoku	33.0%	Х	41.4%	74.4%	Х	Х
Amazons	36.8%	67.9%	60.0%	50.7%	49.0%	Х
Breakthrough	39.0%	Х	69.5%	40.4%	43.9%	48.5%
Santorini	42.7%	Х	59.7%	46.6%	43.3%	Х
Surakarta	33.5%	68.9%	43.1%	35.3%	55.6%	score
Lines of Action	50.9%	Х	57.1%	46.8%	53.7%	44.0%
mean	40.3%	69.1%	54.9%	50.4%	45.4%	46.3%

Table 5: Final winning percentages of the combinations of the experiment of Section 5.3 (X: heuristic without interest in this context; presence coincides with score in Surakarta and Othello; see 3.2; 95% confidence intervals: max  $\pm 1.04\%$ )

of the game tree. In case of a tie,  $UBFM_s$  decides by choosing the one that leads to the state of best value. This decision is safer because the number of times an action is selected is the number of times that this action is more interesting than the others.

**Example 12.** The current player has the choice between two actions  $a_1$  and  $a_2$ . The action  $a_1$  leads to a state of value 5 and was selected 7 times (from the current state and from the beginning of the game). The action  $a_2$  leads to a state of value 2 and was selected 30 times. UBFM chooses the action  $a_1$  while UBFM<sub>s</sub> chooses the action  $a_2$ .

The algorithm of UBFM<sub>s</sub> with completion is described in Algorithm 12 (which uses Algorithm 11; resolution stop is not used for the succinctness of the presentation). *Remark* 13. In some cases, we will prefer to ensure the draw rather than trying to win. Algorithm 12 must then be adapted: to decide the action to play, the first player must then maximize (c(a(s)), r(a(s)), n(s, a), v'(s, a)) and the second player must minimize (c(a(s)), -r(a(s)), n(s, a), v'(s, a)).

#### 6.2. Comparison of Search Algorithms for Game Playing

We now compare the winning percentages of different game algorithms, by using evaluation functions learned by starting again the experiment of Section 4.2 with 16 as repetition number (the used evaluations are those corresponding to the end of the learning process, i.e. h = 48). We compare UBFM<sub>s</sub> with UBFM and iterative deepening alpha-beta with move ordering, denoted ID (each of these algorithms uses completion). For each game, each combination  $(A, f_{\theta_{48}}, r)$  confronts the combinations  $(A', f_{\theta_{48}}, r)$ , with a search time of 1s per action, where  $A \in \{\text{UBFM}_s, \text{UBFM}, \text{ID}\}, A'$  is minimax at depth 1,  $f_{\theta_{48}}$  is one of the final evaluation functions of Section 4.2, and r is the number of one of the 16 repetitions ( $r \in \{1, ..., 16\}$ ). The winning percentage of A is the average of the winning percentage of  $(A, f_{\theta_{48}}, r)$  over the functions  $f_{\theta_{48}}$  and over the repetitions  $r \in \{1, \ldots, 16\}$ . The winning percentages are described in Table 6. For each game, the winning percentage of a search algorithm is calculated from 18432 matches. On all games except Clobber, UBFMs gets the best winning percentage (on two games, Hex and Outer Open Gomoku, UBFMs and ID have the same percentage). On Clobber, it is UBFM which obtains the best percentage, but only 1% more than  $\rm UBFM_s$ . On average, across all games,  $\rm UBFM_s$  is 3.6% better than  $\rm UBFM$  and 5% better than ID.

Then a variation of this experiment was performed. For each game, each combination  $(A, f_{\theta_{48}}, r)$  confronts all the others, but the used evaluation functions  $f_{\theta_{48}}$  are restricted to those generated from the learning algorithm descent and the search time is 10s per action. The corresponding winning percentages are described in Table 7. For each game, the winning percentage of a search algorithm is calculated from 1536 matches. In all games, except Clobber and Santorini, it is again UBFM<sub>s</sub> which obtains the best winning percentage. At Clobber and Santorini, it is UBFM which obtains the best percentage. On average across all games, UBFM<sub>s</sub> is 5% better than UBFM and 19% better than ID. In conclusion, in the context of these experiments, UBFM<sub>s</sub> is the best search algorithm.

*Remark* 14. After the realization of this study, a study comparing UBFM and  $UBFM_s$  to MCTS and ID was been performed at Hex 11 and Hex 13 using evaluation functions

	Outer Open Gomoku	Clobber	Breakthrough	Santorini	Hex
$\rm UBFM_s$	45%	84%	51%	62%	51%
UBFM	38%	85%	45%	58%	41%
ID	45%	82%	45%	55%	51%
	Lines of Action	Othello	Amazons	Surakarta	mean
$\rm UBFM_s$	48%	62%	62%	43%	56.6%
UBFM	41%	61%	61%	38%	52.0%
ID	39%	55%	51%	42%	51.6%

Table 6: Average winning percentages of UBFM<sub>s</sub>, UBFM, and ID over the evaluation functions  $f_{\theta_{48}}$  of Section 4.2, for different games, of the first experiment of Section 6.2 (search time: 1 second per action ; 95% confidence intervals:  $\pm 0.1\%$ ).

	Outer Open Gomoku	Clobber	Breakthrough	Santorini	Hex
UBFM <sub>s</sub>	64%	49%	63%	53%	65%
UBFM	45%	53%	51%	56%	60%
ID	41%	50%	37%	41%	27%
	Lines of Action	Othello	Amazons	Surakarta	mean
UBFM <sub>s</sub>	56%	58%	53%	49%	57%
UBFM	53%	56%	52%	42%	52%
ID	24%	33%	51%	38%	38%

Table 7: Average winning percentages of UBFM<sub>s</sub>, UBFM, and ID over the evaluation functions  $f_{\theta_{48}}$  (generated from descent in Section 4.2), for different games, of the second experiment of Section 6.2 (search time: 10 seconds per action; 95% confidence intervals:  $\pm 0.8\%$ ).

generated from long training runs (more than 30 days) [?]. For each of the performed experiments,  $\rm UBFM_s$  is the best search algorithm, followed by  $\rm UBFM$ .

*Remark* 15. When UBFM is used, to decide which action to play, the equality of values is determined by the number of visits (the most chosen action is preferred).

# 7. Ordinal Distribution and Application to Hex

In this section, we propose the last technique, a new action selection distribution, and we apply it with all the previous techniques to design program-players to the game of Hex.

### 7.1. Ordinal Action Distribution

Thus, we propose an alternative probability distribution (see Section 2.3), that we call *ordinal distribution*. This distribution does not depend on the value of states. However, it depends on the order of their values. Its formula is:

$$P(c_i) = \left(\epsilon' + \frac{1 - \epsilon'}{n - i}\right) \cdot \left(1 - \sum_{j=0}^{j < i} P(c_j)\right)$$

with *n* the number of children of the root,  $i \in \{0, ..., n-1\}$ ,  $c_i$  the *i*-th best child of the root,  $P(c_i)$  the probability of playing the action leading to the child  $c_i$  and  $\epsilon'$ the exploitation parameter ( $\epsilon' = 1 - \epsilon$ ). Algorithm 13 describes the action selection method resulting from the use of the ordinal distribution with an optimized calculation.

*Remark* 16. In an experiment using the common protocol, not presented in this article, ordinal distribution has been mostly better than softmax distribution, but lower than  $\epsilon$ -greedy. However, during long-time learning processes at Hex (similar to the experiments of the following sections), ordinal distribution has performed best.

### 7.2. A Long Training for Hex 11

We now apply all the techniques that we have proposed to carry out a long self-play reinforcement learning on Hex size 11 with swap. More precisely, we use completed descent (Algorithm 9) with tree learning (Algorithm 3), completed ordinal distribution (see Section 5.2 and Algorithm 13), and the additive depth heuristic (see Section 5.1.2).

#### 7.2.1. Technical details

In addition, we use a classical data augmentation: the adding of symmetrical states. Symmetrical states are added in D, the set of pairs (s, v) of the game tree (see Section 3), after the end of each game and before the application of experience replay. Formally,  $D \leftarrow D \cup \{(r_{180^{\circ}}(s), v) \mid (s, v) \in D\}$  where  $r_{180^{\circ}}(s)$  is s rotated by 180°. In other words, the processing(D) method of Algorithm 3 is experience\_replay(symmetry(D),  $(\mu, \sigma)$ ) where symmetry(D) adds symmetrical states in D as described above and returns D. The used learning parameters are: search time per action  $\tau = 2s$ , batch size B = 3000. The experience replay parameters are: games memory size  $\mu = 100^8$ , sampling rate  $\sigma = 5\%$ . We use the following neural network as adaptative evaluation function: a residual network with 4 residual blocks (2 convolutions per block with 83 filters each) followed by a fully connected hidden layers (with 74 neurons). The activation function used is the ReLU. The input of the neural network is a game board extended by one line at the top, bottom, right and left (in the manner of [36, 20]). More precisely, each of these lines is completely filled with the stones of the player of the side where it is located. This extension is simply used to explicitly represent the sides of the board and their membership. Moreover, when the children of a state are evaluated by the neural network, they are batched and thus evaluated in parallel (on the only GPU). The evaluation function has been pre-initialized by learning the values of random terminal states (their number is 15, 168, 000). Other parameters are the same as Section 3.2.1. Resolved states are kept in memory (the memory of the resolved states is emptied every two learning day).

The reinforcement learning process lasted 34, 321 matches. Note that the number of data used during the learning process is of the order of  $59 \cdot 10^7$ , the number of neural network evaluations is of the order of  $196 \cdot 10^6$ , and the number of state evaluations is

<sup>&</sup>lt;sup>8</sup>A variant of replay experience was applied, it is here to memorize the pairs of the last 100 games and not the last 100 pairs.

search time	Mohex 2nd	Mohex 1st	mean	95% confidence interval	total matchs
2.5s	98%	84%	91%	2%	1000
10s	91%	86%	88%	2%	1600

Table 8: Winning percentages against Mohex 3HNN of  $\rm UBFM_s$  using the learned evaluation function of Section 7.2 (the search time per action is the same for each player; default settings are used for Mohex; as many matches in first as in second player).

of the order of  $15 \cdot 10^9$ .

### 7.2.2. Results

The winning percentages against Mohex of  $\rm UBFM_s$  using the learned evaluation function are described in Table 8. Note that the proposed techniques have made it possible to exceed the level of Mohex 3HNN on Hex size 11 (with swap), and without the use of knowledge (which had not been done until then).

### 7.3. A Long Training for Hex 13

We carry out the same experiment as the previous section, but on Hex size 13 (always with swap).

# 7.3.1. Technical differences

The architecture of the neural network is 8 residual blocks (2 convolutions per block with 186 filters each) followed by 2 fully connected hidden layers (with 220 neurons each). The activation function used is the ReLU. The network was not initialized by random end state values. The experience replay parameters are: games memory size  $\mu = 250^9$ , sampling rate  $\sigma = 2\%$ . The search time per action  $\tau$  is 5s. The reinforcement learning process lasted 5, 288 matches. Note that the number of data used during the learning process is of the order of  $16 \cdot 10^7$ , the number of neural network evaluations is of the order of  $64 \cdot 10^6$ , and the number of state evaluations is of the order of  $7 \cdot 10^9$ .

<sup>&</sup>lt;sup>9</sup>A variant of replay experience was applied, it is here to memorize the pairs of the last 250 games and not the last 250 pairs.

search	Mohex 2nd	Mohex 1st	mean	95% confidence interval	total matchs
2.5s	100%	100%	100%	0%	1200
10s	100%	100%	100%	0%	800

Table 9: Winning percentages against Mohex 3HNN of  $\rm UBFM_s$  using the learned evaluation function of Section 7.3 (the search time per action is the same for each player; default settings are used for Mohex; as many matches in first as in second player).

#### 7.3.2. Results

The winning percentages against Mohex 3HNN of  $\rm UBFM_s$  using the learned evaluation function are described in Table 9. The techniques that we have proposed have also allowed for Hex size 13 (with swap) to exceed, the level of Mohex 3HNN, without the use of knowledge (which had also not been done until then).

### 8. Conclusion

We have proposed several new techniques for reinforcement learning state evaluation functions.

Firstly, we have generalized tree bootstrapping (tree learning) and shown that learning the values of the game tree instead of just learning the value of the root significantly improves learning performances with (iterative deepening) alpha-beta, and that this combination is more efficient than MCTS with tree learning, root learning or terminal learning, in the context of reinforcement learning without knowledge based on non-linear functions.

Secondly, we have introduced the descent algorithm which explores in the manner of unbounded best-first minimax, intended to be used during the learning process. Unlike the latter, descent iteratively explores the sequences of best actions *up to the terminal states*. Its objective is to improve the quality of the data used during the learning phases, while keeping the advantages of unbounded best-first minimax. In the context of our experiments, the use of descent gives significantly better performances than the use of alpha-beta, unbound minimax, or MCTS.

Thirdly, we have suggested to replace the classic gain of a game (+1/-1) by

a different terminal evaluation function. We have propose different general terminal evaluations, such as the depth heuristic, which takes into account the duration of games in order to favor quick wins and slow defeats. Our experiments have shown that the use of a reinforcement heuristic improves performances and that the depth heuristic is a useful general heuristic.

Fourthly, we have proposed another variant of best-first minimax, which plays the safest action instead of playing the best action. It is intended to be used, in particular, after the learning process. In the experiments carried out, it is this algorithm which obtains the best winning percentages on average and in 8 of the 9 tested games. We have also proposed a new action selection distribution which does not take into account the value of states but only their order.

In the context of this experiments, the techniques that we propose are the best approach of reinforcement learning of state values. We apply all these different techniques to design program-players to Hex with swap (size 11 and 13) surpassing, for the first time, the level of Mohex 3HNN (the best Hex program based on supervised learning and dedicated Hex algorithms and knowledge) by reinforcement learning from self-play without knowledge (or Monte Carlo techniques).

Overall, the results show that unbounded minimax, which has obtained good results in the experiments of this article, has been under-studied in the literature and that unbounded minimax and its variants, that we offer, seems to constitute particularly efficient alternatives to alpha-beta and MCTS, in the era of reinforcement learning.

A promising research perspective is the parallelization of UBFM and *descent* when learning: UBFM determines the next action to play and *descent* determines the pairs to learn. The other research perspectives include the application of our contributions to the game of Go and to General Game Playing (at first with perfect information). They also include the application and adaptation of our contributions to the contexts of hidden information, stochastic and multiplayer games. Moreover, they include the application of our contributions to optimization problems, such that the RNA Inverse Folding problem [63].

Remark 17. The realized programs (based on the descent framework) are programmed

in Python. Switching to a faster performing language should reduce the learning time by a factor between two and five.

Note to conclude that, on the one hand, the descent framework with the unbounded minimax faced Polygames during the computer olympiad of 2020 and beat it at Othello 8x8, Othello 10x10 and Breakthrough. In fact, 5 gold medals were won by the proposed algorithms for the following games: Othello 10x10, Breakthrough, Surakarta, Amazons, and Clobber. At Clobber and at Surakarta, one of the participating programs was also based on AlphaZero. That of Clobber used in addition the combinatorial game theory to solve states. Moreover, a silver medal were won by the proposed algorithms at Othello 8x8. It is the first time at the Computer Olympiad that the same program wins 5 gold medals.

On the other hand, the descent framework has also competed in the 2021 Computer Olympiad. This time it won 11 gold medals (Hex  $11 \times 11$ , Hex  $13 \times 13$ , Hex  $19 \times 19$ , Havannah  $8 \times 8$ , Havannah  $10 \times 10$ , Othello  $8 \times 8$ , Surakarta, Amazons, Breakthrough, Brazilian Draughts, Canadian Draughts; there was no competition at Othello 10x10 and Clobber). Descent notably beat Polygames at games where they met (Hex  $13 \times 13$ , Hex  $19 \times 19$ , Havannah  $8 \times 8$ , Havannah  $10 \times 10$ ).

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Algorithm 9 Descent tree search algorithm with completion and resolution stop (see Table 1 for the definitions of symbols and Algorithm 10 for the definitions of completed\_best\_action(s) and backup\_resolution(s)). Note: T = (v, v', c, r) and S is the set of states of the partial game tree which are non-leaves or terminal.

```
Function completed_descent_iteration (s, S, T, f_{\theta}, f_{t})
    if terminal (s) then
         S \leftarrow S \cup \{s\}
         r(s), c(s), v(s) \leftarrow 1, \mathbf{b}_{\mathbf{t}}(s), f_{\mathbf{t}}(s)
     else
         if s \notin S then
              S \leftarrow S \cup \{s\}
              foreach a \in actions(s) do
                   if terminal (a(s)) then
                        S \leftarrow S \cup \{a(s)\}
                        v'(s, a) \leftarrow f_t(a(s))
                       r(a(s)), c(a(s)), v(a(s)) \leftarrow 1, b_t(a(s)), v'(s, a)
                   else
                  | v'(s,a) \leftarrow f_{\theta}(a(s))
              a_b \leftarrow \texttt{completed_best_action}(s, \texttt{actions}(s))
              c(s), v(s) \leftarrow c(a_b(s)), v'(s, a_b)
              r(s) \leftarrow \text{backup}_\text{resolution}(s)
         if r(s) = 0 then
              A \leftarrow \{a \in \operatorname{actions} (s) \mid r(a(s)) = 0\}
              a_b \leftarrow \text{completed\_best\_action\_dual}(s, A)
              n(s, a_b) \leftarrow n(s, a_b) + 1
              v'(s, a_b) \leftarrow \text{completed\_descent\_iteration}(a_b(s))
              a_b \leftarrow \text{completed\_best\_action}(s, \text{actions}(s))
              c(s), v(s) \leftarrow c(a_b(s)), v'(s, a_b)
              r(s) \leftarrow \text{backup}_{\text{resolution}}(s)
    return v(s)
```

Function completed\_descent (s, S, T,  $f_{\theta}$ ,  $f_{t}$ ,  $\tau$ ) t = time()while time() - t <  $\tau \land r(s) = 0$  do completed\_descent\_iteration(s, S, T,  $f_{\theta}$ ,  $f_{t}$ ) return S, T 51 Algorithm 10 Definition of the algorithms completed\_best\_action(s, A), which computes the *a priori* best action by using completion, and backup\_resolution(s), which updates the resolution of s from its child states.

**Function** completed\_best\_action (*s*, *A*)

 $\begin{array}{c|c} \text{if first_player}\left(s\right) \text{ then} \\ & | \quad \operatorname{return} \mathop{\arg\max}_{a \in A} \left(c\left(a\left(s\right)\right), v'\left(s,a\right), n\left(s,s'\right)\right) \\ \text{else} \\ & | \quad \operatorname{return} \mathop{\arg\min}_{a \in A} \left(c\left(a\left(s\right)\right), v'\left(s,a\right), -n\left(s,s'\right)\right) \end{array}$ 

Function completed\_best\_action\_dual(s, A)

 $\begin{array}{c|c} \text{if first_player}\left(s\right) \text{ then} \\ | & \operatorname{return} \arg \max \left(c\left(a\left(s\right)\right), v'\left(s,a\right), -n\left(s,s'\right)\right) \\ \text{else} \\ | & \operatorname{return} \arg \min \left(c\left(a\left(s\right)\right), v'\left(s,a\right), n\left(s,s'\right)\right) \\ & a \in A \end{array} \right)$ 

**Function** backup\_resolution(s)

if |c(s)| = 1 then | return 1 else | return min  $a \in actions(s) r(a(s))$  Algorithm 11 UBFM<sub>s</sub> tree search algorithm with completion and resolution stop (see Table 1 for the definitions of symbols and Algorithm 10 for the definitions of completed\_best\_action(s) and backup\_resolution(s)). Note: T = (v, v', c, r, n).

**Function** ubfms\_iteration (s, S, T,  $f_{\theta}$ ,  $f_{t}$ ) if terminal (s) then  $S \leftarrow S \cup \{s\}$  $r(s), c(s), v(s) \leftarrow 1, b_t(s), f_t(s)$ else if r(s) = 0 then if  $s \notin S$  then  $S \leftarrow S \cup \{s\}$ foreach  $a \in actions(s)$  do if terminal (a(s)) then  $S \leftarrow S \cup \{a(s)\}$  $v'(s,a) \leftarrow f_{t}\left(a(s)\right)$  $r(a(s)), c(a(s)), v(a(s)) \leftarrow 1, \mathbf{b}_{\mathbf{t}}(a(s)), v'(s, a)$ else  $| v'(s,a) \leftarrow f_{\theta}(a(s))$ else  $A \leftarrow \{a \in \operatorname{actions} (s) \mid r(a(s)) = 0\}$  $a_b \leftarrow \texttt{completed\_best\_action\_dual}(s, A)$  $n(s, a_b) \leftarrow n(s, a_b) + 1$  $v'(s, a_b) \leftarrow \texttt{ubfms_iteration}(a_b(s))$  $a_b \leftarrow \text{completed\_best\_action}(s, \text{actions}(s))$  $c(s), v(s) \leftarrow c(a_b(s)), v'(s, a_b)$  $r(s) \leftarrow \text{backup}_{resolution}(s)$ return v(s)

**Function** ubfms\_tree\_search (s, S, T,  $f_{\theta}$ ,  $f_{t}$ ,  $\tau$ )

t= time () while time ()  $-t<\tau\wedge r\left(s\right)=0$  do ubfms\_iteration (s, S, T, f\_{\theta}, f\_{\rm t}) return S,T Algorithm 12 UBFM<sub>s</sub> action decision algorithm with completion (see Table 1 for the definitions of symbols). Note: T = (v, v', c, r, n) and S is the set of states of the game tree which are non-leaves or terminal. Function safest\_action (s, T)if first\_player (s) then return  $\underset{a \in actions(s)}{\operatorname{arg\,max}} (c(a(s)), n(s, a), v'(s, a))$ else return  $\underset{a \in actions(s)}{\operatorname{arg\,min}} (c(a(s)), -n(s, a), v'(s, a))$ Function ubfms  $(s, S, T, f_{\theta}, f_{t}, \tau)$   $S, T \leftarrow$  ubfms\_tree\_search  $(s, S, T, f_{\theta}, f_{t})$ return safest\_action (s, T)

**Algorithm 13** Ordinal action distribution algorithm with simulated annealing used in the experiments of this article (see Table 1 for the definitions of symbols).

 Function ordinal (s, v')

 if first\_player (s) then

 |  $A \leftarrow actions (s)$  sorted in descending order by  $a \mapsto v'(s, a)$  

 else

 |  $A \leftarrow actions (s)$  sorted in ascending order by  $a \mapsto v'(s, a)$ 
 $j \leftarrow 0$ 
 $n \leftarrow |A|$  

 for  $a \in A$  do

 | if probability  $(\frac{t}{T} \cdot (n - j - 1) + 1) / (n - j)$  then

 |  $j \leftarrow j + 1$